Geometric uncertainty relation, the symplectic area, and the J-holomorphic maps for mixed quantum states

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Abstract

In this paper we will establish a relation between geometric uncertainty relation and the determinant of the quantum covariance matrix for mixed quantum states. We will show that determinant of the covariance matrix represents the squared metric area of a parallelogram. In this setting the geometric uncertainty relation compares a metric area to a symplectic area. Moreover, we will in details investigate relation between J-holomorphic maps and geometric uncertainty relation for mixed quantum states. We will argue that determinant of the quantum covariance matrix is equal to the harmonic energy of a holomorphic map that minimize the areas.

1 Introduction

In recent years, we have also witness the birth of symplectic topology where the study of global structures of a symplectic manifold is in the main focus. One of the important result of this field is the Gromov's non-squeezing theorem which has found application in geometric quantum mechanics. In [1], the author suggest that the Gromov's non-squeezing theorem is a classical version of the geometric uncertainty relation for a pure quantum state. Moreover, de Gosson and Luef [2] have related the quantum covariance matrix and geometric uncertainty relation for pure state to the symplectic capacitance and Gromov's non-squeezing theorem. Recently we have introduced a geometric framework for general for mixed quantum states based on a specific Kähler structure. Using this geometric framework we were able to derived a geometric uncertainty relation for mixed quantum state [3].
Now, in this paper, we will explore the relation between the geometric uncertainty relation and the new results from symplectic topology for mixed quantum states. First we define the determinant of quantum covariance matrix and then we show that this matrix represents the squared metric area of a parallelogram. Next we show that the determinant of quantum covariance

matrix is equal to the harmonic energy of a map $u : \Sigma \longrightarrow \mathcal{H}$, where a Σ is a two dimensional surface immersed in the Hilbert space. We also investigate the geometric uncertainty relation in terms of J-holomorphic map [4]. In particular in section 2 we will review our geometric framework for mixed quantum states. Next, in section 3 we will study the geometric uncertainty relation and quantum covariance matrix determinant. Then in section 4 we will establish a relation between the quantum covariance matrix determinant and a symplectic area spanned by two vector fields. Moreover, in section 5 we will give a short introduction to J-holomorphic maps. Furthermore in section 6 we will discuss the minimal geometric uncertainty relation and the holomorphic condition. We will also illustrate the theory covered in the paper by an example in section 7.

2 Geometric framework

Here we will review our geometric framework for mixed quantum states based on a Kähler structure [3]. A density operator ρ on the Hilbert space $\mathcal{H}$ is a member of the space $\mathrm{Her}(\mathcal{H})$ of Hermitian operators on $\mathcal{H}$ whose eigenvalues are non-negative and sum up to 1. Let $\mathcal{D}(\mathcal{H})$ be the space of density operators on $\mathcal{H}$. Two density operators belong to the same orbit if and only if they have the same spectrum. Given such a spectrum σ, we write $\mathcal{D}(\sigma)$ for the corresponding orbit. In this section we introduce an Ad-equivariant Kähler structure on $\mathcal{D}(\sigma)$ called the Kirillov-Kostant-Souriau Kähler structure [5]. For each density operator ρ we have a surjective linear map

$$\Lambda_\rho : \mathrm{Her}(\mathcal{H}) \to \mathrm{T}_\rho\mathcal{D}(\sigma) \tag{2.0.1}$$

defined by

$$\Lambda_\rho(\hat{H}) = \frac{1}{i\hbar}[\hat{H}, \rho] \tag{2.0.2}$$

since any elements in $\mathrm{T}_\rho\mathcal{D}(\sigma)$ can be written as $[\hat{H}, \rho]$. The kernel of Λ_ρ consists of all Hermitian operators on $\mathcal{H}$ that commutes with ρ, and we define a complementary space to $\operatorname{Ker}\Lambda_\rho$ as follows.

Let $p_1 > p_2 > \cdots > p_k$ be the eigenvalues of the density operator, σ, and n_j be the multiplicity of p_j. We can always find a basis in $\mathcal{H}$ relative which

$$\rho = \begin{pmatrix} p_1\mathbf{1}_{n_1} & 0 & \cdots & 0 \\ 0 & p_1\mathbf{1}_{n_2} & 0 & \vdots \\ \vdots & \cdots & \ddots & 0 \\ 0 & \cdots & 0 & p_1\mathbf{1}_{n_k} \end{pmatrix}, \tag{2.0.3}$$

where $\mathbf{1}_{n_i}$ for $i = 1, 2, \ldots k$ are $n_i \times n_i$ identity matrices. Moreover, the kernel of Λ_ρ consists of all those Hermitian operators $\hat{A}$ which are represented by

block diagonal matrices

$$\hat{A} = \operatorname{diag}(A_{11}, A_{22}, \dots, A_{kk}), \tag{2.0.4}$$

relative to this basis where each A_{ii} is an $n_i \times n_i$ Hermitian matrix. The complementary space $\operatorname{Ker}\Lambda_\rho^\perp$ is defined to consist of all the Hermitian operators that are represented by off-diagonal matrices

$$\hat{B} = \begin{bmatrix} \mathbf{0}_{n_1} & B_{12} & B_{13} & \dots & B_{1k} \\ B_{12}^\dagger & \mathbf{0}_{n_2} & B_{23} & \dots & B_{2k} \\ B_{13}^\dagger & B_{23}^\dagger & \mathbf{0}_{n_3} & \dots & B_{3k} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ B_{1k}^\dagger & B_{2k}^\dagger & B_{3k}^\dagger & \dots & \mathbf{0}_{n_k} \end{bmatrix}. \tag{2.0.5}$$

Thus $\operatorname{Her}(\mathcal{H}) = \operatorname{Ker}\Lambda_\rho \oplus \operatorname{Ker}\Lambda_\rho^\perp$, and Λ_ρ maps $\operatorname{Ker}\Lambda_\rho^\perp$ isomorphically onto the tangent bundle of the quantum phase space $\mathrm{T}_\rho\mathcal{D}(\sigma)$. Next we will define an almost complex structure on the quantum phase space.
An almost complex structure on the space $\mathcal{D}(\sigma)$ is an automorphism of its tangent bundle whose square equals $-\mathbf{1}$ that is, the bundle map $J : \mathrm{T}\mathcal{D}(\sigma) \to \mathrm{T}\mathcal{D}(\sigma)$, defined by

$$J\left(\frac{1}{i\hbar}[\hat{B}, \rho]\right) = \frac{1}{i\hbar}[j(\hat{B}), \rho], \tag{2.0.6}$$

where, e.g., for the matrix $\hat{B} = (B_{kl})$ we have $j(\hat{B}) = (iB_{kl})$ satisfies $J^2 = -\mathbf{1}$.

3 Geometric uncertainty relation and quantum covariance matrix determinant

In this section we review our construction of the Kirillov-Kostant-Souriau symplectic form on the quantum phase space $\mathcal{D}(\sigma)$. We also discuss the construction of our geometric uncertainty relation for mixed quantum states based on this geometric framework. Finally we define the determinant of the quantum covariance matrix in terms of the geometric uncertainty relation. The Kirillov-Kostant-Souriau symplectic form on $\mathcal{D}(\sigma)$ is defined by

$$\omega\left(\frac{1}{i\hbar}[\hat{A}, \rho], \frac{1}{i\hbar}[\hat{B}, \rho]\right) = \frac{1}{i\hbar}\operatorname{Tr}\left([\hat{A}, \hat{B}]\rho\right) = \frac{1}{i\hbar}\operatorname{Tr}\left(\hat{A}[\hat{B}, \rho]\right). \tag{3.0.7}$$

Note that, the symplectic form ω (3.0.7) is non-degenerated and closed. The importance of this form stems from the fact that if A is the expectation value function of a Hermitian operator $\hat{A}$, that is $A(\rho) = \operatorname{Tr}(\rho\hat{A})$, and X_A is the

Hamiltonian vector field associated with A, which is implicitly defined by the identity $dA(X) = \omega(X_A, X)$, then

$$X_A(\rho) = \frac{1}{i\hbar}[\hat{A}, \rho]. \tag{3.0.8}$$

Theorem 3.0.1 *Let $\hat{A}$ and $\hat{B}$ be two observables on the Hilbert space which are off-diagonal at ρ. Then we have*

$$h(X_A(\rho), X_B(\rho)) = \sum_{i>j}(p_i - p_j)\operatorname{Tr}(A_{ij}^{\dagger}B_{ij}),$$

where A_{ij} and B_{ij} are elements of $\hat{A}$ and $\hat{B}$ respectively.

The proof of this theorem can be found in our recent work [3]. The above result is very important in proof of a geometric uncertainty relation for mixed quantum states which we will consider now.
Let $\hat{A}$ be a observable on $\mathcal{H}$, and consider the uncertainty function

$$\Delta A(\rho) = \sqrt{\operatorname{Tr}(\rho\hat{A}^2) - \operatorname{Tr}(\rho\hat{A})^2}. \tag{3.0.9}$$

The proof of the following theorem also can be found in [3].

Theorem 3.0.2 *Let $\hat{A}$ and $\hat{B}$ be two obervables on $\mathcal{H}$. Then we have*

$$\Delta A(\rho)\Delta B(\rho) \geq |h(X_A(\rho), X_B(\rho))|. \tag{3.0.10}$$

Note that

$$|h(X_A(\rho), X_B(\rho))|^2 = g(X_A(\rho), X_B(\rho))^2 + \omega(X_A(\rho), X_B(\rho))^2, \tag{3.0.11}$$

where

$$g(X_A(\rho), X_B(\rho)) = \operatorname{Re} h(X_A(\rho), X_B(\rho)) = \operatorname{Re}\sum_{i>j}(p_i - p_j)\operatorname{Tr}(A_{ij}^{\dagger}B_{ij}) \tag{3.0.12}$$

and

$$\omega(X_A(\rho), X_B(\rho)) = \operatorname{Im} h(X_A(\rho), X_B(\rho)) = \operatorname{Im}\sum_{i>j}(p_i - p_j)\operatorname{Tr}(A_{ij}^{\dagger}B_{ij})$$

On the other hand we have

$$\begin{aligned}(\Delta A(\rho))^2(\Delta B(\rho))^2 &\geq |h(X_A(\rho), X_B(\rho))|^2 \\ &= \big(g(X_A(\rho), X_B(\rho))^2 + \omega(X_A(\rho), X_B(\rho))^2\big)\end{aligned}$$

which can be written as

$$(\Delta A(\rho))^2(\Delta B(\rho))^2 - g(X_A(\rho), X_B(\rho))^2 \quad \geq \quad \omega(X_A(\rho), X_B(\rho))^2$$

Now let us define a quantum covariance matrix determinant as follows

$$\begin{aligned} D(X_A(\rho), X_B(\rho)) &= (\Delta A(\rho))^2(\Delta B(\rho))^2 - g(X_A(\rho), X_B(\rho))^2 \\ &\geq \omega(X_A(\rho), X_B(\rho))^2. \end{aligned} \tag{3.0.13}$$

In the next section we will use this result to establish a relation between the geometric uncertainty relation and a symplectic area.

4 The quantum covariance matrix determinant and the symplectic area

We will show that the quantum covariance matrix determinant is also equal to the determinant of the Jacobian matrix of an immersion map of a Riemannian surface Σ into $\mathcal{D}(\sigma)$.
Let ρ vary over $\mathcal{D}(\sigma)$ and define the following vector fields

$$v, w : \mathcal{D}(\sigma) \longrightarrow T\mathcal{D}(\sigma) \tag{4.0.14}$$

such that $v(\rho) = [\hat{A}, \rho] = i\hbar X_A(\rho)$ and $w(\rho) = [\hat{B}, \rho] = i\hbar X_B(\rho)$.

Lemma 4.0.3 *Let $\hat{A}$ and $\hat{B}$ be two hermitian linear operator on $\mathcal{D}(\sigma)$ and let $\rho \in \mathcal{D}(\sigma)$. Then the covariance matrix determinant*

$$\begin{aligned} D(X_A(\rho), X_B(\rho)) &= \det \begin{pmatrix} (\Delta A(\rho))^2 & g(X_A(\rho), X_B(\rho)) \\ g(X_B(\rho), X_A(\rho)) & (\Delta B(\rho))^2 \end{pmatrix} \\ &= (\Delta A(\rho))^2(\Delta B(\rho))^2 - g(X_A(\rho), X_B(\rho))^2 \end{aligned}$$

represents the square area of a parallelogram defined by vectors $v(\rho)$ and $w(\rho)$ defined above measured in metric g in $\mathcal{D}(\sigma)$, that is

$$\begin{aligned} D(X_A(\rho), X_B(\rho)) &\geq g(X_A(\rho), X_A(\rho))g(X_B(\rho), X_B(\rho)) \\ &- g(X_A(\rho), X_B(\rho))g(X_B(\rho), X_A(\rho)). \end{aligned}$$

Proof 4.0.4 *First we note that $g(X_A(\rho), X_A(\rho)) \leq (\Delta A(\rho))^2$ and*

$$g(X_B(\rho), X_B(\rho)) \leq (\Delta B(\rho))^2.$$

Moreover, we have $g(X_A(\rho), X_B(\rho)) = g(X_B(\rho), X_A(\rho))$. Now, we know that a formula for the area of a parallelogram is defined by two vectors v and w which are elements of a hermitian vector space with the following scalar form $h(\cdot,\cdot) = g(\cdot,\cdot)+i\omega(\cdot,\cdot)$. In this case the metric area of the parallelogram

is $\mathcal{A} = |v||w||\sin\theta|$, where θ denotes the angle between two vectors v and w. Thus we could write the square of the area as follows

$$\begin{aligned}
\mathcal{A}^2 &= |v|^2|w|^2 - |v|^2|w|^2\cos^2\theta && (4.0.15)\\
&= g(v,v)g(w,w) - g(v,w)g(w,v). && (4.0.16)
\end{aligned}$$

Now we realize that the equation (4.0.15) represents the square area of a parallelogram defined by the vectors $X_A(\rho)$ and $X_B(\rho)$ in $\mathcal{D}(\sigma)$.

Proposition 4.0.5 *The geometric uncertainty relation establishes a relation between the metric and symplectic differential areas defined by vectors $v(\rho)$ and $w(\rho)$*

$$\sqrt{D(X_A(\rho), X_B(\rho))} \;\geq\; \omega(X_A(\rho), X_B(\rho)). \qquad (4.0.17)$$

Proof 4.0.6 *The symplectic form $\omega(X_A(\rho), X_B(\rho))$ measures an area in $\mathcal{D}(\sigma)$ since it is a volume form in two dimensions. Thus by the above lemma, the geometric uncertainty relation in the form (7.0.38) provides us with a relation between the two 2-dimensional differential areas.*

5 J-holomorphic maps

Here we will give a short introduction to the J-holomorphic maps. Consider a Riemanian surface Σ with $\dim\Sigma = 2$, e.g. $\Sigma = \mathbb{S}^2$. Moreover let (Σ, j) and $(\mathcal{M}, J)$ be two almost complex manifolds. The the map

$$u : (\Sigma, j) \longrightarrow (\mathcal{M}, J)$$

satisfies the J-holomorphic condition

$$J \circ du = du \circ j, \qquad (5.0.18)$$

where du is a vector-valued one-form with values in $u^*T(\mathcal{M})$ and can be decomposed into J-holomorphic and J-anti-holomorphic parts

$$du = \partial_J u + \bar{\partial}_J u, \qquad (5.0.19)$$

such that

$$\partial_J u = \frac{1}{2}(du - J \circ du \circ j), \qquad (5.0.20)$$

and

$$\bar{\partial}_J u = \frac{1}{2}(du + J \circ du \circ j). \qquad (5.0.21)$$

Hence the map u is J-holomorphic if and only if $\bar{\partial}_J u = 0$. The harmonic energy of a smooth map u is defined as the L^2-norm of the one-form $du \in \Lambda^1(\Sigma, u^*T(\mathcal{M}))$ by

$$E(u) \quad = \quad \frac{1}{2}\int_\Sigma |du|_g^2 dvol_\Sigma, \tag{5.0.22}$$

where in coordinate systems $(x_1, x_2) \in \Sigma$ and $(x_1, \ldots, x_{2n}) \in \mathcal{M}$ we could write $g = \sum g_{\alpha\beta} dy_\alpha dy_\beta$ and $h = \sum h_{ij} dx_i dx_j$ with $(h^{ij}) = (h_{ij})^{-1}$, so we have

$$|du|_g^2 \quad = \quad \sum_{i,j,\alpha,\beta} g_{\alpha\beta}(u(x)h^{ij}(x)\frac{\partial u_\alpha}{\partial x^i}\frac{\partial u_\beta}{\partial x^j}. \tag{5.0.23}$$

The map u could be a J-holomorphic curve in a symplectic manifold. In this case the harmonic energy of u is a topological invariant which is dependent only on the homology class of the curve modulo some possible boundary. The proof of the following lemma and corollary can be found in [4, 1].

Lemma 5.0.7 *Let $(\mathcal{M}, \omega)$ be a symplectic manifold equipped with a compatible almost complex structure J. Then every smooth map $u : \Sigma \longrightarrow \mathcal{M}$ satisfies*

$$\begin{aligned} E(u) \quad &= \quad \int_\Sigma |\bar{\partial}_J(u)|_g^2 dvol_\Sigma + \int_\Sigma u^*\omega \\ &\geq \quad \int_\Sigma u^*\omega \end{aligned} \tag{5.0.24}$$

and we have equality whenever u is J-holomorphic. If Σ is a closed surface without boundary, then $\int_\Sigma u^*\omega$ is constant in fixed homology class. In such case, if we have a J-holomorphic map, then the metric area $\mathcal{A}_{g_J}(u) = E(u)$ depends only on the homology class of u which we denote by $[u]$, that is

$$\mathcal{A}_{g_J}(u) = [\omega]([u]). \tag{5.0.25}$$

Corollary 5.0.8 *Assume that $u : \Sigma \longrightarrow \mathcal{M}$ is J-holomorphic map, then in neighborhood of each regular point of u on Σ, the image of u, is a minimal surface with respect to the metric $g_J(\cdot,\cdot) = \omega(\cdot, J\cdot)$.*

In the following section we will apply these results to quantum phase space.

6 Minimal geometric uncertainty relation and the holomorphic condition

In this section we investigate the relation between minimal geometric uncertainty and the holomorphic condition for mixed quantum states. Note

that a complex structure means that the manifold has coordinates that are complex-valued and with holomorphic transition functions. This also implies that they locally look like $\mathbb{C}^n$, both geometrically and analytically. Let Σ be an open subset of $\mathbb{C}$ which is immersed in $\mathcal{D}(\sigma)$, or $\mathcal{H} \cong \mathbb{C}^m = \mathbb{R}^{2m}$. Then

$$u : ((0,1) \times (0,1), j) \longrightarrow (\mathcal{D}(\sigma), J) \tag{6.0.26}$$

defined by $(s,t) \longmapsto u(s,t) = sv + tw$ where $v, w \in \mathcal{D}(\sigma)$. We rewrite these vector field as

$$v = (v^1, \ldots, v^{2m})^T, \quad w = (w^1, \ldots, w^{2m})^T. \tag{6.0.27}$$

Then we have

$$\begin{aligned} u(s,t) &= (y^1, \ldots, y^{2m})^T \\ &= (sv^1 + tw^1, \ldots, su^{2m} + tw^{2m})^T \end{aligned} \tag{6.0.28}$$

and $u^*(dy^\alpha) = v^\alpha ds + w^\alpha dt$, with $\alpha = 1, 2, \ldots, 2m$. Thus du become the matrix of partial derivatives with columns v and w

$$du = \begin{pmatrix} \frac{\partial y^1}{\partial s} & \frac{\partial y^1}{\partial t} \\ \vdots & \vdots \\ \frac{\partial y^{2m}}{\partial s} & \frac{\partial y^{2m}}{\partial t} \end{pmatrix} = \begin{pmatrix} v^1 & w^1 \\ \vdots & \vdots \\ v^{2m} & w^{2m} \end{pmatrix} \tag{6.0.29}$$

on Σ and

$$\begin{aligned} h &= u^*G = u^*\left(\sum_\alpha (dy^\alpha)^2\right) = \sum_\alpha (u^*(dy^\alpha))^2 \\ &= |v|^2 ds^2 + |w|^2 dt^2 + 2\langle v, w\rangle ds dt \\ &= h_{ij} ds_i dt_j \end{aligned} \tag{6.0.30}$$

which gives $h_{11} = |v|^2$, $h_{22} = |w|^2$ and $h_{12} = h_{21} = \langle v, w\rangle$. Thus we have

$$\begin{aligned} |du|^2 &= \sum_{i,j,\alpha} h^{ij}(s,t)\left(\frac{\partial^\alpha y_\alpha}{\partial s}\frac{\partial^\alpha y_\alpha}{\partial t}\right)_{ij} \\ &= (\det(h_{ij}))^{-1}(2|v|^2|w|^2 - 2\langle v, w\rangle^2) = 2. \end{aligned} \tag{6.0.31}$$

Now, the harmonic energy is given by

$$\begin{aligned} E(u) &= \int_\Sigma dvol_\Sigma = \int_\Sigma \sqrt{\det(h_{ij})} ds \wedge dt \\ &= \sqrt{|v|^2|w|^2 - \langle v, w\rangle^2}. \end{aligned} \tag{6.0.32}$$

Thus we have shown the following lemma.

Lemma 6.0.9 *The harmonic energy of the immersion u is equal to the area of the parallelogram*

$$E(u) \;=\; \sqrt{D(X_A(\rho), X_B(\rho))} = \sqrt{(\Delta A(\rho))^2(\Delta B(\rho))^2 - g(X_A(\rho), X_B(\rho))^2}$$

We have also the following important lemma on J-holomorphic map.

Lemma 6.0.10 *The map $u : ((0,1)\times(0,1), j) \longrightarrow (\mathcal{D}(\sigma), J)$ is J-holomorphic if and only if $v_\alpha + iv_{\alpha+1} = -i(w_\alpha + iw_{\alpha+1})$, thus if and only if $v = -Jw$.*

Proof 6.0.11 *First we choose our basis as $(e_1 + ie_1, e_2 + ie_2, \ldots e_n + ie_n)$. Then we obtain the J-holomorphic condition $\bar{\partial}_J u = 0$, where J is a $2m \times 2m$ block matrix where each block is equal to $j = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ as follows.*

$$\begin{aligned}
\bar{\partial}_J u &= \frac{1}{2}(du + J \circ du \circ j)) \qquad (6.0.33)\\
&= \frac{1}{2}\left(\begin{pmatrix} v_1 & w_1 \\ v_2 & w_2 \\ v_3 & w_3 \\ v_4 & w_4 \\ \vdots & \vdots \end{pmatrix} + \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ & & \ddots & \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} v_1 & w_1 \\ v_2 & w_2 \\ v_3 & w_3 \\ v_4 & w_4 \\ \vdots & \vdots \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\right)\\
&= \frac{1}{2}\left(\begin{pmatrix} v_1 & w_1 \\ v_2 & w_2 \\ v_3 & w_3 \\ v_4 & w_4 \\ \vdots & \vdots \end{pmatrix} + \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ & & \ddots & \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} w_1 & -v_1 \\ w_2 & -v_2 \\ w_3 & -v_3 \\ w_4 & -v_4 \\ \vdots & \vdots \end{pmatrix}\right)\\
&= \frac{1}{2}\left(\begin{pmatrix} v_1 & w_1 \\ v_2 & w_2 \\ v_3 & w_3 \\ v_4 & w_4 \\ \vdots & \vdots \end{pmatrix} + \begin{pmatrix} -w_2 & v_2 \\ w_1 & -v_1 \\ -w_4 & v_4 \\ w_3 & -v_3 \\ \vdots & \vdots \end{pmatrix}\right) = \frac{1}{2}\begin{pmatrix} v_1 - w_2 & w_1 + v_2 \\ v_2 + w_1 & w_2 - v_1 \\ v_3 - w_4 & w_3 + v_4 \\ v_4 + w_3 & w_4 - v_3 \\ \vdots & \vdots \end{pmatrix}\\
&= 0
\end{aligned}$$

Thus the map u is J-holomorphic if and only if $v_\alpha + iv_{\alpha+1} = -i(w_\alpha + iw_{\alpha+1})$, which is equivalent to $v = -Jw$.

Note that off-diagonal part of metric $g(X_A(\rho), X_B(\rho))$ vanishes in case $v = -Jw$. Thus we have the following corollary.

Corollary 6.0.12 *If the map u is J-holomorphic, then the metric $g(X_A(\rho), X_B(\rho))$ has vanishing off-diagonal elements and minimum determinant.*

Now, if we put all together, then we arrive at the following theorem.

Theorem 6.0.13 *The harmonic energy $E(u)$, where the map defined in (6.0.26) is equal to $\sqrt{D(X_A(\rho), X_B(\rho))}$ with*

$$v, w : \mathcal{D}(\sigma) \longrightarrow T\mathcal{D}(\sigma) \tag{6.0.34}$$

such that $v(\rho) = [\hat{A}, \rho] = i\hbar X_A(\rho)$ and $w(\rho) = [\hat{B}, \rho] = i\hbar X_B(\rho)$. Moreover, the harmonic energy $E(u)$ represents the area of the image of the open unit square under the map u. Furthermore, the geometric uncertainty relation in integral form is given by

$$\begin{aligned} E(u) &= \frac{1}{2}\int_\Sigma |du)|_g^2 ds \wedge dt \\ &\geq \int_\Sigma u^*\omega, \end{aligned} \tag{6.0.35}$$

We have equality in this integral formulation if the off-diagonal part of the metric $g(X_A(\rho), X_B(\rho))$ vanishes and then we get

$$\Delta A(\rho)\Delta B(\rho) = \int_\Sigma u^*\omega \tag{6.0.36}$$

which is a topological invariant within a fixed homology class of the curve.

In the following section we will give an example on a quantum two-level systems that illustrate this results.

7 Example

Consider a qubit realized as a physical system when the spin up polarization is p_1 and the proportion with spin down polarization is p_2, and let $\mathbf{S}$ be the spin-$\frac{1}{2}$ operator. If we model the spin part of the system on $\mathbb{C}^2$ in such a way that $|\uparrow\rangle$ and $|\downarrow\rangle$ represent the spin up and spin down states, respectively, then the state of the spin part of the ensemble can be represented by the density operator $\rho = \begin{pmatrix} p_1 & 0 \\ 0 & p_2 \end{pmatrix}$ and we have

$$\hat{S}_x = \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \hat{S}_y = \frac{\hbar}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \hat{S}_z = \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

If $p_1 \neq p_2$, then we have that

$$\{S_x, S_y\}_g(\rho) = 2\hbar \operatorname{Re}\operatorname{Tr}\begin{pmatrix} ip_1/4 & 0 \\ 0 & -ip_2/4 \end{pmatrix} = 0.$$

Moreover, we have

$$\{S_x, S_y\}_\omega(\rho) = 2\hbar \operatorname{Im}\operatorname{Tr}\begin{pmatrix} ip_1/4 & 0 \\ 0 & -ip_2/4 \end{pmatrix} = \frac{\hbar}{2}(p_1 - p_2).$$

Thus the off-diagonal terms of the covariance matrix vanishes and we have

$$\begin{aligned}\sqrt{D(\hat{S}_x(\rho), \hat{S}_y(\rho))} &= \Delta S_x(\rho)\Delta S_y(\rho) \\ &\geq \omega(\hat{S}_x(\rho), \hat{S}_y(\rho)) = \frac{\hbar}{2}(p_1 - p_2)\end{aligned} \tag{7.0.37}$$

Moreover, the harmonic energy of the immersion u is given by

$$E(u) = \sqrt{D(\hat{S}_x(\rho), \hat{S}_y(\rho))} \geq \frac{\hbar}{2}(p_1 - p_2). \tag{7.0.38}$$

Thus the variance $\Delta S_x(\rho)\Delta S_y(\rho)$ is a topological invariant, with a fixed homology class of a curve. Moreover, the map u is a J-holomorphic map and the unit square is mapped to a square. In this example the squared root of quantum covariance matrix is greater than the symplectic area which is equal to the half of the Planck constant multiplied with the difference between the eigenvalues of the density matrix. Thus it is proportional to the Planck constant as in the case of a pure state. We have also gave an explicit expression for the harmonic energy of the map u.

8 Conclusion

In this paper we have established a relation between geometric uncertainty relation, determinant of the quantum covariance matrix, and J-holomorphic maps. We have shown that the quantum covariance matrix is equal to the sysmplectic area spanned by two vector fields on the quantum phase space. We have also investigated the relation between the harmonic energy $E(u)$ of a immersed map u into the quantum phase space and determinant of the quantum covariance matrix. In particular we have argued that for the case of a J-holomorphic map the harmonic energy is equal to the product of the dispersion function of two observables on the quantum phase space. However, this result needs further investigation.

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